Uniformly continuous maps between ends of $\mathbb{R}$-trees

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Abstract

There is a well-known correspondence between infinite trees and ultrametric spaces which can be interpreted as an equivalence of categories and comes from considering the end space of the tree.

In this equivalence, uniformly continuous maps between the end spaces are translated to some classes of coarse maps (or even classes of metrically proper lipschitz maps) between the trees.

Keywords: Tree, ultrametric, end space, coarse map, uniformly continuous, non expanding map.

MSC: Primary: 54E35; 53C23 Secondary: 54C05; 51K05

1 Introduction

This paper is mainly inspired by a recent, interesting and beautiful one due to Bruce Hughes [4] but it is also motivated by [8] where a complete ultrametric was defined on the sets of shape morphisms between compacta.

In [8] it was proved that every shape morphism induces a uniformly continuous map between the corresponding ultrametric spaces of shape morphisms which are, in particular, complete and bounded as metric spaces. Moreover Hughes established some categorical equivalences for some classes of ultrametric spaces and local similarity equivalences to certain categories of geodesically complete rooted $\mathbb{R}$-trees and certain equivalence classes of isometries at infinity.

In view of that, it is natural for us to ask for a description of uniform types (the classification by means of uniform homeomorphism) of end spaces of geodesically complete rooted $\mathbb{R}$-trees in terms of some geometrical properties of the trees.

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To answer these questions is the aim of this paper and we find herein that the bounded coarse geometry, see [10] and [11], of $\mathbb{R}$-trees is an adequate framework to do that.

Also, we would like to point out some important differences between this paper and Hughes's. First of all we treat different, although related, categories:

The morphisms in every category of ultrametric spaces used in [4] are isomorphisms for the uniform category of ultrametric spaces, i.e. they are uniformly continuous homeomorphisms, while in this paper we get results for the whole category of complete bounded ultrametric spaces and uniformly continuous maps between them (not only for uniformly continuous homeomorphism).

But above all, we get an explicit formula to construct a non-expansive map between two trees that induces a given uniformly continuous function between the corresponding end spaces. To obtain this formula we use a procedure described by Borsuk, [3], to find a suitable modulus of continuity associated to a uniformly continuous function. This is the way in which we pass from the total disconnectedness of ultrametric spaces to the strong connectivity of any ray in the tree.

Our main results in this paper can be summarized as follows:

The category of complete ultrametric spaces with diameter bounded above by 1 and uniformly continuous maps between them is isomorphic to any of the following categories:

1) Geodesically complete rooted $\mathbb{R}$-trees and metrically proper homotopy classes of metrically proper continuous maps between them.

2) Geodesically complete rooted $\mathbb{R}$-trees and coarse homotopy classes of coarse continuous maps between them.

3) Geodesically complete rooted $\mathbb{R}$-trees and metrically proper non-expansive homotopy classes of metrically proper non expansive continuous maps between them.

We finish this paper recovering, as a consequence of our constructions, the classical relation between the proper homotopy type of a locally finite simplicial tree and the topological type of its Freudenthal end space, see [1].

Although our main source of information on $\mathbb{R}$-trees is Hughes's paper [4], it must be also recommended the classical book [12] of Serre and the survey [2] of Bestvina for more information and to go further, let us say that in [7], J. Morgan treats a generalization of $\mathbb{R}$-trees called Λ-trees. Moreover, in [5], Hughes and Ranicki treat applications of ends, not only ends of trees, to topology.

2 Trees

We are going to recall some basic properties on trees mainly extracted from [4].

Definition 2.1. *A* real tree*, or* $\mathbb{R}$-tree *is a metric space* (T, d) *that is uniquely arcwise connected and* $\forall x, y \in T$*, the unique arc from* x *to* y*, denoted* $[x, y]$*, is isometric to the subinterval* $[0, d(x, y)]$ *of* $\mathbb{R}$*.*

Lemma 2.2. *If* T *is an* $\mathbb{R}$*-tree and* $v, w, z \in T$*, then there exists* $x \in T$ *such that* $[v, w] \cap [v, z] = [v, x]$*.*

Definition 2.3. *A* rooted $\mathbb{R}$-tree*,* (T, v) *is an* $\mathbb{R}$*-tree (T,d) and a point* $v \in T$ *called* the root*.*

Definition 2.4. *A rooted* $\mathbb{R}$*-tree is* geodesically complete *if every isometric embedding* $f : [0, t] \to T$*,* $t > 0$*, with* $f(0) = v$*, extends to an isometric embedding* $\tilde{f} : [0, \infty) \to T$*. In that case we say that* $[v, f(t)]$ *can be extended to a* geodesic ray*.*

Remark 2.5. *The single point* v *is a trivial rooted geodesically complete* $\mathbb{R}$*-tree.*

Notation: If (T, v) is a rooted $\mathbb{R}$-tree and $x \in T$, let $\|x\| = d(v, x)$,

$$B(v, r) = \{x \in T | \ \|x\| < r\}$$
$$\bar{B}(v, r) = \{x \in T | \ \|x\| \leq r\}$$
$$\partial B(v, r) = \{x \in T | \ \|x\| = r\}$$

Example 2.6. *Cantor tree. Assume that each edge of the tree has length 1.*

Example 2.7. $\{(x, y) \in \mathbb{R}^2 | x \geq 0$ *and* $y = 0, y = x$ *or* $y = \frac{x}{2^n}$ *with* $n \in \mathbb{N}\}$

For any two points on different branches x,y define $d(x, y) = d_u(x, v) + d_u(v, y)$*.*

Example 2.8. *Consider* $(\mathbb{R}^2, O)$ *and for any two points non aligned with the origin define the distance as follows* $d((x_1, x_2), (y_1, y_2)) = \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2}$*.*

Definition 2.9. *If* c *is any point of the rooted* $\mathbb{R}$*-tree* (T, v)*, the* subtree of (T, v) determined by c *is:*

$$T_c = \{x \in T | \ c \in [v, x]\}.$$

Also, let

$$T_c^i = T_c \backslash \{c\} = \{x \in T | \ c \in [v, x] \wedge x \neq c\}.$$

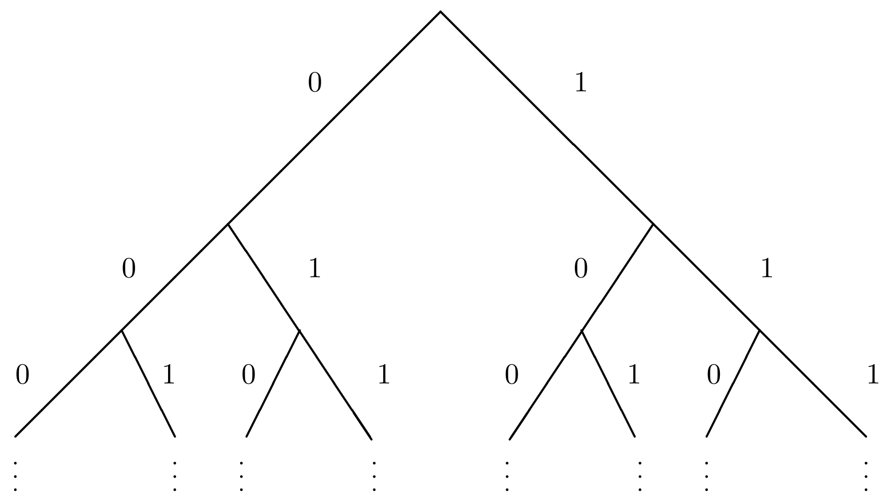

Figure 1: The Cantor tree.

Lemma 2.10. *If (T, v) is a geodesically complete rooted $\mathbb{R}$-tree, T_c the subtree induced by any point c and $x \in (T, v)$ such that $x \notin T_c$ then $\forall y \in T_c \quad d(x, y) = d(x, c) + d(c, y)$.*

Proof. It suffices to show that $c \in [x, y]$. Lemma 2.2 implies that there exists $z \in (T, v)$ such that $[v, x] \cap [v, y] = [v, z]$ and we start with $x \notin T_c$, that is, $c \notin [v, x]$, in particular, $c \notin [v, z]$ and $c \in [z, y]$. It is clear that $[x, y] = [x, z] \cup [z, y]$ thus, $c \in [x, y]$. $\square$

Lemma 2.11. *Let (T, v) a geodesically complete rooted $\mathbb{R}$-tree, T_c the subtree induced by c and $x \in (T, v)$ such that $x \notin T_c$ then $d(x, T_c) = d(x, c)$.*

Proof. It follows immediately from 2.10. $\square$

Remark 2.12. *Let c be any point of the geodesically complete rooted $\mathbb{R}$-tree (T, v), then T_c is closed.*

Let $x \notin T_c$ and $\epsilon = d(x, T_c) = d(x, c) > 0$. By 2.10, $B(x, \epsilon) \cap T_c = \emptyset$. Hence $T \backslash T_c$ is open.

Remark 2.13. *Let c be any point of the geodesically complete rooted $\mathbb{R}$-tree (T, v), then T_c^i is open.*

Let $x \in T_c^i$ ($c \in [v, x]$ and $x \neq c$) and $\epsilon = d(x, c) > 0$. Then $B(x, \epsilon) \subset T_c^i$ and hence, T_c^i is open.

Lemma 2.14. *If $f : [0, \infty) \to (T, v)$ is an isometric embedding such that $f(0) = v$, then $\forall t_0 \in [0, \infty) \qquad f[t_0, \infty) \subset T_{f(t_0)}$.*

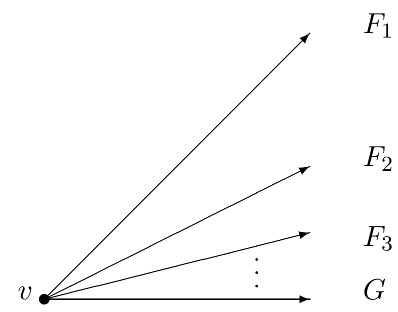

Figure 2: Non locally finite tree.

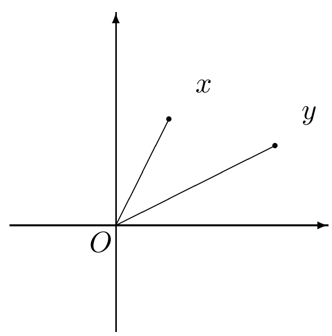

Figure 3: $\mathbb{R}$–tree which is not a $\mathbb{Z}$–tree.

Proof. Clearly $\forall t > t_0$, $f(t_0)$ must be in $[v, f(t)]$. Hence $f(t) \in T_{f(t_0)}$. $\square$

Remark 2.15. *Let $c \in (T, v)$ a geodesically complete rooted $\mathbb{R}$-tree, then T_c is also a geodesically complete rooted $\mathbb{R}$-tree.*

T_c is a metric space since it is a subset of a metric space. It is clear that any point in T_c is connected with c by an arc, so, any two points in T_c are connected by an arc which is obviously unique since T_c is a subset of (T, v) which is uniquely arcwise connected.

We take c as the root of T_c.

Let $f : [0, t_0] \to T_c$ any isometric embedding such that $f(0) = c$. We consider the isometric embedding $f' : [0, t_0 + \|c\|] \to T$ such that $f'(0) = v$, $f'(\|c\|) = c$ and $f'(t + \|c\|) = f(t)$. f' extends f and, by definition of geodesically complete, there exists an isometric embedding $\tilde{f}'[0, \infty) \to T$ such that $\tilde{f}'$ extends f'. $\tilde{f}'(\|c\|) = c$ and by lemma 2.14 $\tilde{f}'[\|c\|, \infty) \subset T_c$. If we define $\tilde{f}(t) = \tilde{f}'(t + \|c\|)$ it is readily seen that $\tilde{f} : [0, \infty) \to T_c$ is an isometric embedding and extends f in T_c.

Definition 2.16. *A* cut set *for a geodesically complete rooted $\mathbb{R}$-tree (T,v) is a subset C of (T,v) such that $v \notin C$ and for every isometric embedding $\alpha : [0,\infty) \to T$ with $\alpha(0) = v$ there exists a unique $t_0 > 0$ such that $\alpha(t_0) \in C$.*

Example 2.17. *$\partial B(v,r)$ with $r > 0$, is a* cut set *for (T,v).*

Proposition 2.18. *Given a cut set C for (T,v), the connected components of $T(C) := \{x \in T|\ [v,x] \cap C \neq \emptyset\}$ (that is, the part of (T,v) not between the root and the cut set) are exactly the subtrees $\{T_c\}_{c\in C}$.*

Proof. $T(C) = \underset{c\in C}{\cup} T_c$ and we know that T_c is always connected (as it is in fact arcwise connected). Let's see that for any $c_0 \in C$, the connected component of c_0 in $T(C)$ is T_{c_0}.

If we remove from the tree any point $x \in (T,v)$ we disconnect the tree in two subsets: T_x^i and $T\backslash T_x$ which are open sets in (T,v), as we saw in lemmas 2.12 and 2.13, and it is easy to verify that T_x^i is clopen in $T\backslash\{x\}$. (Note that T_x^i need not be connected but we may remark that T_x^i is a union of connected components of an open set and these are open since (T,v) is locally connected).

Let $c' \in C$ such that $c' \neq c_0$ and $w \in (T,v)$ such that $[v,c_0] \cap [v,c'] = [v,w]$. Consider $x \in [w,c_0]$ such that $x \neq c_0$ and by definition of cut set it is clear that $x \notin T(C)$ and $T_x^i \cap T(C)$ is a clopen set in $T(C)$ that contains T_{c_0} and $T_x^i \cap T_{c'} = \emptyset$. The intersection of all the clopen sets that contain T_{c_0} (we already know that T_{c_0} is connected), is the quasi-component of T_{c_0} which contains the connected component and doesn't intersect any other subtree $T_{c'}$ induced by any other point of the cut set. Hence, the connected component of c_0 is exactly T_{c_0}. $\square$

Remark 2.19. *If we consider the cut set in (T,v) $C := \partial B(v,r)$ with $r > 0$, then $T(C)$ is exactly $T\backslash B(v,r)$.*

3 Metrically proper maps between trees

Here we used [10] and [11] for the main concepts.

Definition 3.1. *A map f between two metric spaces X, X' is* metrically proper *if for any bounded set A in X', $\quad f^{-1}(A)$ is bounded in X.*

Definition 3.2. *A map f between two rooted $\mathbb{R}$-trees, $f : (T,v) \to (T',w)$, is said to be* rooted *if $f(v) = w$.*

To avoid repeating the expression: rooted, continuous and metrically proper map we define metrically proper between trees as follows.

Definition 3.3. *A map f between two rooted R-trees is* metrically proper between trees *if it is rooted, metrically proper and continuous.*

Remark 3.4. *If $f:(T,v)\to(T',w)$ is a metrically proper map between trees, then:*

$$\forall M>0 \quad \exists N>0 \text{ such that } \quad f^{-1}(B(w,M))\subset B(v,N).$$

This is equivalent to say that $f(T\backslash B(v,N))\subset T'\backslash B(w,M)$.

Proposition 3.5. *Let $f:(T,v)\to(T',w)$ be a metrically proper map between trees, and let $M>0$ and $N>0$ such that $f^{-1}(B(w,M))\subset B(v,N)$, then*

$$\forall c\in\ \partial B(v,N)\ \exists!\ c'\in\partial B(w,M) \text{ such that } \quad f(T_c)\subset T'_{c'}.$$

Proof. Let $f:T\to T'$ be a metrically proper map between trees, then $\forall M>0 \quad \exists N>0$ such that $\quad f^{-1}(B(w,M))\subset B(v,N)\Longrightarrow f(T\backslash B(v,N))\subset T'\backslash B(w,M)$. f sends connected components of $T\backslash B(v,N)$ into connected components of $T'\backslash B(w,M)$.

In particular, $\forall c\in\ \partial B(v,N) \quad f(T_c)\subset T'\backslash B(w,M)$. As it is a continuous image of a connected set, is clearly contained in one of the connected components of $T'\backslash B(w,M)$, and those are, as we saw in proposition 2.18 and 2.17, the subtrees determined by points of the cut set $\partial B(w,M)$. $\square$

Equivalence relation on metrically proper maps between trees

In this paragraph we introduce an equivalence relation and the resulting equivalence classes form the morphisms of the category whose objets are geodesically complete rooted $\mathbb{R}$-trees. We are going to define this equivalence relation in two steps, first we are going to put it in terms of maps restricted to complements of closed balls centered at the root, and using that, we will demonstrate that the relation is in fact a metrically proper homotopy. The interest of this equivalence class is that two maps will be in the same class if and only if they induce the same map between the end spaces of the trees (that will be uniformly continuous as we shall see).

Let $M>0,\ N>0$ be such that $f(T\backslash B(v,N))\subset T'\backslash B(w,M)$ and $\forall c\in\partial B(v,N)$ let T_c the subtree determined by c. By proposition 3.5, $\exists! c'\in\partial B(w,M)$ such that $f(T_c)\subset T_{c'}$.

This allows us to consider a map which sends the subtrees of $T\backslash B(v,N)$ to subtrees of $T'\backslash B(w,M)$ as follows.

Definition 3.6. *Given $\mathcal{T}_N:=\{T_c|\ c\in\partial B(v,N)\}$, let $f_{\mathcal{T}_N}:\mathcal{T}_N\longrightarrow \mathcal{T}'_M$ such that $f_{\mathcal{T}_N}(T_c)=T_{c'}\Leftrightarrow f(T_c)\subset T_{c'}$.*

This map can be defined from a certain N_0 (which depends on M), and for all $N>N_0$. If $N>N_0$, then $\forall d\in\partial B(v,N)$ there exists a unique $c\in\partial B(v,N_0)$ such that $T_d\subset T_c$, and obviously

$$f(T_d)\subset f(T_c)\subset T'_{c'}\Rightarrow f_{\mathcal{T}_{N'}}(T_d)=T'_{c'}.$$

Definition 3.7. *Given $f, f' : (T, v) \to (T', w)$ two metrically proper maps between trees, then*

$$f \sim f' \Leftrightarrow \forall M > 0, \quad \exists N_0 > 0 \text{ such that } \quad \forall N > N_0 \quad f_{\mathcal{T}_N} = f'_{\mathcal{T}_N}.$$

Proposition 3.8. *$\sim$ defines an equivalence relation.*

Proof. It is obviously reflexive and symmetric.

Transitive: If $f \sim f'$ and $f' \sim f''$ then there exists N_0 such that $\forall N > N_0 \quad f_{\mathcal{T}_N} = f'_{\mathcal{T}_N}$ and exists N_1 such that $\forall N > N_1 \quad f'_{\mathcal{T}_N} = f''_{\mathcal{T}_N}$ hence, for all $N > max\{N_0, N_1\}$ we may check that $f \sim f''$. □

Definition 3.9. *If $f, g : X \to T$ are two continuous maps from any topological space X to a tree T then the* shortest path homotopy *is an homotopy $H : X \times I \to T$ of f to g such that if $j_x : [0, d(f(x), g(x))] \to [f(x), g(x)]$ is the isometric immersion of the subinterval $[0, d(f(x), g(x))] \subset \mathbb{R}$ into T whose image is the shortest path between $f(x)$ and $g(x)$, then $H(x,t) = j_x(t \cdot d(f(x), g(x)))$ $\forall t \in I$ $\forall x \in X$.*

Lemma 3.10. *If $f, g : X \to T$ are two continuous maps from any topological space $(X,\mathcal{T})$ to a tree T then there is a shortest path homotopy, $H : X \times I \to T$ of f to g.*

Proof. It suffices to prove that H with the definition above is continuous. Consider $(x_0, t_0) \in X \times I$. The continuity of f and g implies that $\forall \epsilon > 0$ there exists $x_0 \in U \in \mathcal{T}$ such that $f(U) \subset B_T(f(x_0), \frac{\epsilon}{2})$ and $g(U) \subset B_T(g(x_0), \frac{\epsilon}{2})$. It is immediate to check that this implies that $H(U, t_0) \subset B_T(H(x_0, t_0), \frac{\epsilon}{2})$. Let K be such that $d(f(x), g(x)) < K \quad \forall x \in U$. Then, $H(U, B(t_0, \frac{\epsilon}{2K})) \subset B_T(H(x_0, t_0), \epsilon)$ and H is continuous. Clearly, $H_0 \equiv f$ and $H_1 \equiv g$. □

Definition 3.11. *Given $f, f' : (T, v) \to (T', w)$ two metrically proper maps between trees, let H be a continuous map $H : T \times I \to T'$ with $H(v,t) = w$ $\forall t \in I$ such that $\forall M > 0, \exists N > 0$ such that $H^{-1}(B(v, M)) \subset B(v, N) \times I$. H is a rooted metrically proper homotopy of f to f' if $H|_{T \times \{0\}} = f$ and $H|_{T \times \{1\}} = f'$.*

Notation: $f \simeq_{Mp} f'$ if and only if there exists a rooted metrically proper homotopy of f to f'.

Definition 3.12. *Two trees (T, v),(T', w) are metrically properly homotopic, $(T \simeq_{Mp} T')$, if and only if there exist two metrically proper maps between trees $f : T \to T'$ and $f' : T' \to T$, such that $f \circ f' \simeq_{Mp} id_{T'}$ and $f' \circ f \simeq_{Mp} id_T$.*

Proposition 3.13. *$f \sim f' \Leftrightarrow f \simeq_{Mp} f'$.*

Proof. Suppose $f \sim f'$. $\forall n \in \mathbb{N}$ let $t_n > 0$ such that $f(T \backslash B(v, t_n)) \subset T' \backslash B(w, n)$ and $f'(T \backslash B(v, t_n)) \subset T' \backslash B(w, n)$. Without loss of generality suppose $t_{n+1} > t_n + 1$. If $f \sim f'$ by proposition 3.5 $\forall c$ in the cut set $\partial B(v, t_n)$ in T, there exists a unique point c' in the cut set $\partial B(w, n)$ in T', such that the image under either f or f', of T_c, is contained in $T'_{c'}$.

By 3.10 if we consider the shortest path homotopy of f to f' it remains to check that this homotopy is metrically proper. It suffices to show that $\forall t_n$ and $\forall t \in [0, 1]$ $H_t(T \backslash B(v, t_n)) \subset T' \backslash B(w, n)$. Given $x \in T \backslash B(v, t_n)$ we know that $f(x) \in T' \backslash B(w, n)$ and $f'(x) \in T' \backslash B(w, n)$ and also, by definition 3.7, $\exists! c' \in \partial B(w, n)$ such that $f(x) \in T'_{c'}$ and $f'(x) \in T'_{c'}$. As we saw in remark 2.15, $T'_{c'}$ is an $\mathbb{R}$-tree, so there exists an arc in that tree from $f(x)$ to $f'(x)$, and, since T is uniquely arcwise connected, this arc must be the same and must be contained in $T'_{c'}$. Hence the homotopy restricted to $T \backslash B(v, t_n)$ is contained in $T' \backslash B(w, n)$.

Conversely, given $f, f' : (T, v) \to (T', w)$ consider $H : T \times I \to T'$ a metrically proper homotopy of f to f'. Let $M > 0$, $N > 0$ such that $H_t(T \backslash B(v, N)) \subset T' \backslash B(w, M)$ $\forall t \in I$. For any $c \in \partial B(v, N)$ and $c' \in \partial B(w, M)$ such that $f(T_c) \subset T'_{c'}$ it is clear that $H_t(T_c) \subset T'_{c'}$ $\forall t \in I$ (as it is the continuous image of a connected set into $T' \backslash B(w, M)$), and in particular (if $t = 1$), $f'(T_c) \subset T'_{c'}$, and hence, $f \sim f'$. $\square$

4 Ultrametric spaces

We include in this section the definition and some elementary properties of ultrametric spaces. Most of these properties are not going to be needed through this paper but we believe that they are very helpful to imagine the structure of an ultrametric space.

Definition 4.1. *If (X, d) is a metric space and $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$, then d is an* ultrametric *and (X, d) is an* ultrametric *space.*

Lemma 4.2. *(a) Any point of a ball is a center of the ball.*

(b) If two balls have a common point, one is contained in the other.

(c) The diameter of a ball is less than or equal to its radius.

(d) In an ultrametric space, all triangles are isosceles with at most one short side.

(e) $S_r(a) = \bigcup_{x \in S_r(a)} B_{<r}(x)$.

(f) The spheres $S_r(a)$ $(r > 0)$ are both open and closed.

All these properties are demonstrated and beautifully exposed in [9].

5 The end space of a tree

In this section we define the functor ξ from trees to ultrametric spaces following step by step [4].

Definition 5.1. *The* end space *of a rooted* $\mathbb{R}$*-tree* (T, v) *is given by:*

$$end(T, v) = \{f : [0, \infty) \to T \mid f(0) = v \textit{ and } f \textit{ is an isometric embedding } \}.$$

For $f, g \in end(T, v)$*, define:*

$$d_e(f, g) = \begin{cases} 0 & \textit{if } f = g, \\ e^{-t_0} & \textit{if } f \neq g \textit{ and } t_0 = sup\{t \geq 0 |\ f(t) = g(t)\} \end{cases}$$

Note that since T is uniquely arcwise connected:

$$\{t \geq 0 |\ f(t) = g(t)\} = \begin{cases} [0, \infty) \text{ if } f = g, \\ [0, t_0] \text{ if } f \neq g. \end{cases}$$

Proposition 5.2. *If* (T, v) *is a rooted* $\mathbb{R}$*-tree, then* $(end(T, v), d_e)$ *is a complete ultrametric space of diameter* ≤ 1.

Proposition 5.3. *For any* $x \in (T, v)$*, a geodesically complete rooted* $\mathbb{R}$*-tree, there exist* $F \in end(T, v)$ *and* $t \in [0, \infty)$ *such that* $F(t) = x$ *(in fact,* $t = \|x\|$*).*

Proof. $[0, d(v, x)] = [0, \|x\|] \approx$ [1] $[v, x]$ and by 2.4, it extends to a geodesic ray $F = \{f : [0, \infty) \to T |\ f \text{ isometry } \}$. The result is a geodesic ray (an element of the end space of the tree), F, such that $F(\|x\|) = x$. □

6 The tree of an ultrametric space

Definition 6.1. *Let* U *a complete ultrametric space with diameter* ≤ 1*, define:*

$$T_U := \frac{U \times [0, \infty)}{\sim}$$

with $(\alpha, t) \sim (\beta, t') \Leftrightarrow t = t'$ and $\alpha, \beta \in U$ such that $d(\alpha, \beta) \leq e^{-t}$.

Given two points in T_U represented by equivalence classes $[x, t], [y, s]$ with $(x, t), (y, s) \in U \times [0, \infty)$ define a metric on T_U by:

$$D([x, t], [y, s]) = \begin{cases} |t - s| & \text{if } x = y, \\ t + s - 2\min\{-ln(d(x, y)), t, s\} & \text{if } x \neq y. \end{cases}$$

[1] isometry

Remark 6.2. *Instead of defining the tree as in [4] for any ultrametric space of finite diameter we restrict ourselves to ultrametric spaces of diameter* ≤ 1*. We place the root in* $[(x, 0)]$ *and thus the ultrametric space is isometric to the end space of the tree.*

Proposition 6.3. D *is a metric on* T_U*.*

Proposition 6.4. (T_U, D) *is a geodesically complete rooted* $\mathbb{R}$*-tree.*

Proposition 6.5. $U \approx end(T_U)$*.*

Proof. Consider the map $\gamma : U \to end(T_U)$ which sends each $\alpha \in U$ to the isometric embedding $f_\alpha : [0, \infty) \to T_U$ such that $f_\alpha(t) = (\alpha, t)$ ($f_\alpha \in end(T_U)$).

Given $\alpha, \beta \in U$ let $d_0 = d(\alpha, \beta)$ then $(\alpha, t) = (\beta, t)$ on $[0, -ln(d_0)]$ and in the end space, $d(f_\alpha, f_\beta) = e^{ln(d_0)} = d_0$ and hence, γ is an isometry. It is immediate to see that it is surjective by the completeness of U. $\square$

7 Constructing the functors

7.1 Maps between trees induced by an uniformly continuous map between the end spaces

The purpose in this section is, from a uniformly continuous map between two ultrametric spaces (with diameter ≤ 1) to induce a map between the trees of these spaces. As we have seen, the spaces are isometric to the end spaces of their trees, so we can suppose that uniformly continuous map directly between the ends.

Definition 7.1.1. *A function* $\varrho : [0, \infty) \longrightarrow [0, \infty)$ *is called* modulus of continuity *if* ϱ *is non-decreasing, continuous at* 0 *and* $\varrho(0) = 0$*.*

Lemma 7.1.2. *Let* (X_1, d_1)*,* (X_2, d_2) *two metric spaces,* X_2 *bounded and let* $f : X_1 \to X_2$ *a uniformly continuous map. Then* $\exists \varrho : [0, \infty) \to [0, \infty)$ *modulus of continuity such that* $\forall x, y \in X_1 \quad d_2(f(x), f(y)) \leq \varrho(d_1(x, y))$*.*

Proof. Define:

$$\varrho(\delta) := \sup_{x,y \in X_1,, \ d(x,y) \leq \delta} \{d(f(x), f(y))\}. \tag{1}$$

We are going to show that ϱ is a modulus of continuity. ϱ is well defined since X_2 is bounded, and it is immediate that is non-decreasing and $\varrho(0) = 0$. It remains to check the continuity at 0. Since f is uniformly continuous, then $\forall \varepsilon > 0 \quad \exists\ \delta > 0$ such that $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$ then $\varrho(\delta') \leq \varepsilon\ \forall \delta' < \delta$, and hence,

$$\lim_{\delta \to 0} \varrho(\delta) = 0.$$

$\square$

To define the map between the trees we will need the modulus of continuity to be continuous, so to define the functor, it suffices to show that for the map between the end spaces, there exists a continuous modulus of continuity such as in lemma 7.1.2.

We only need to show that there exists such a map to define the functor and to prove the categorical equivalence, nevertheless, we can construct (1) and in certain examples, to follow the process from 7.1.3 to 7.1.5 and see what λ exactly does, and so, for simple examples, we can get an analytic expression of this map between the trees as we shall see later on in 7.1.13.

Following the construction of Borsuk in [3], we take something similar to a convex hull of the image to obtain a continuous (and convex) modulus of continuity.

Definition 7.1.3. *$\forall x \in [0, \infty)$ let $\Gamma(x)$ the set of ordered pairs (x_1, x_2) such that $\quad x_1, x_2 \in [0, \infty),\ x_1 < x_2$ and $\quad x \in [x_1, x_2]$.*

Definition 7.1.4. *If $x \in [x_1, x_2] \quad \exists!\ t \in [0, 1]$ such that $\quad x = tx_1 + (1 - t)x_2$. Let $\varrho_{x_1,x_2}(x) = t\varrho(x_1) + (1 - t)\varrho(x_2)$*

Definition 7.1.5.

$$\omega(x) := \sup_{x_1,x_2 \in \Gamma(x)} \varrho_{x_1,x_2}(x)$$

Proposition 7.1.6. *$\omega(0) = 0$ and $\omega(x)$ is increasing, convex, uniformly continuous and*

$$\lim_{x \to 0} \omega(x) = 0$$

Proof. It is clear that $\omega(0) = \varrho(0) = 0$. It is immediate to see that it is increasing since ϱ is, and convex obviously by construction. The proof that it is continuous at 0 is in [3]. $\square$

Remark 7.1.7. *Note that by definition $\omega(x) \geq \varrho(x) \quad \forall x \in [0, \infty)$.*

The ultrametric spaces we are considering are of diameter ≤ 1 so, we may assume $im(\omega) \subset [0, 1]$. Define $\lambda := \omega|_{[0,1]}$ and suppose $\lambda(1) = 1$.

There is no loss of generality since if $\lambda(1) < 1$ we can find another convex map, greater or equal than this one, with the same properties and such that its image of 1 is 1. It suffices to define $\varrho'(1) = 1$ and $\varrho'(t) = \lambda(t)\ \forall t \in [0, 1)$. From this map, we rewrite the process to construct a convex map ω' as in 7.1.3, 7.1.4 and 7.1.5, that will be greater or equal than ϱ' and $\omega'(1) = 1$. We consider the restriction to $[0, 1]$ and so we get the map λ' that we were looking for.

Hence, from a uniformly continuous map f between two ultrametric spaces U_1, U_2 with diameter $(U_i) \leq 1$, we get a map $\lambda : [0, 1] \to [0, 1]$ uniformly continuous, convex and non-decreasing such that:

$$\lim_{\delta \to 0} \lambda(\delta) = 0$$

with $\lambda(0)=0,\ \lambda(1)=1$ and by remark 7.1.7:

$$\forall x,\ y\in U_1 \quad d(f(x),f(y))\leq \lambda(d(x,y)).$$

Using this map we are now in position to induce from a uniformly continuous map f, between two complete ultrametric spaces of diameter ≤ 1, a map between the induced trees. As we saw in 6.5, we can identify this ultrametric spaces with the end spaces of the trees and given $f:U_1\to U_2$ a uniformly continuous map, by abuse of notation consider $f: end(T_{U_1},v)\to end(T_{U_2},w)$ such that for any $x\in U_1$, the isometric embedding whose image is $x\times[0,\infty)(\in end(T_{U_1},v))$ is sent to the isometric embedding whose image is $f(x)\times[0,\infty)(\in end(T_{U_2},w))$, transforming f into a map between end spaces.

Definition 7.1.8. *Let (T,v),(T',w) two geodesically complete rooted $\mathbb{R}$-trees, and let $f: end(T,v)\to end(T',w)$ a uniformly continuous map. Then define $\hat{f}: T\to T'$ such that $\forall x\in T$ let $F\in end(T,v),\ t\in[0,\infty)$ with $x=F(t)$ then $\hat{f}(x)=f(F)\Big(-ln(\lambda(e^{-t}))\Big)$.*

Remark 7.1.9. $-ln(\lambda(e^{-t}))$ *is non-decreasing.*

$t_1>t_0\Rightarrow e^{-t_1}<e^{-t_0}\Rightarrow \lambda(e^{-t_1})\leq\lambda(e^{-t_0})\Rightarrow -ln(\lambda(e^{-t_1}))\geq -ln(\lambda(e^{-t_0}))$.

Moreover, if $d_0=min\{d>0|\ \lambda(d)=1\}$, then λ is strictly increasing on $[0,d_0]$ since it is convex, and hence, it is immediate to check that $-ln(\lambda(e^{-t}))$ is strictly increasing for t on $[-ln(d_o),\infty)$. This implies that the map $\hat{f}|_{F[-ln(d_0),\infty)}$ will be injective for every $F\in end(T,v)$.

Remark 7.1.10. *Note that*

$$\lim_{\delta\to 0}\lambda(\delta)=0\Rightarrow \lim_{t\to\infty}\Big(-ln(\lambda(e^{-t}))\Big)=\infty.$$

Now we are going to verify that this map is well-defined and then we shall study its properties.

Well defined Each point in (T,v) has a unique image.

Let $x\in(T,v)$, and $F,G\in end(T,v)$, $t_0,t_1\in[0,\infty)$ such that $F(t_0)=x=G(t_1)$. We already know that $t_0=t_1$ and $F(t)=G(t)\ \ \forall t\in[0,t_0]\Rightarrow d(F,G)\leq e^{-t_0}$ and by (1) $\Rightarrow d(f(F),f(G))\leq\lambda(e^{-t_0})$.

This means that the branches of the tree (the image of the isometric embeddings of $[0,\infty)$) $f(F)$ and $f(G)$ coincide at least until the image of x and so, the image of x is unique.

Now, $d(f(F),f(G))=e^{-sup\{s\geq 0/f(F)(s)=f(G)(s)\}}\leq\lambda(e^{-t_0})\Leftrightarrow sup\{s\geq 0/f(F)(s)=f(G)(s)\}\geq -ln(\lambda(e^{-t_0}))$ and in particular $f(F)\Big(-ln(\lambda(e^{-t_0}))\Big)=f(G)\Big(-ln(\lambda(e^{-t_0}))\Big)$. Since the image by $\hat{f}$ doesn't depend on the representative, is well defined.

Proposition 7.1.11. *If f is a uniformly continuous map between the end spaces then $\hat{f}$, is* ***Lipschitz of constant 1****.*

Proof. Given $x, x' \in (T, v)$, we are going to prove that $d(\hat{f}(x), \hat{f}(x')) \leq d(x, x')$:

Case I. If the points are in the same branch of the tree.

Then, there exists $F \in end(T, v)$ such that $x = F(t_0)$ and $x' = F(t_1)$ with $t_1 > t_0$, and hence $d(x, x') = t_1 - t_0$.

The images are $f(F)\Big(-ln(\lambda(e^{-t_0}))\Big)$ and $f(F)\Big(-ln(\lambda(e^{-t_1}))\Big)$ and it is clear that

$$d(\hat{f}(x), \hat{f}(x')) = \Big| -ln\Big(\lambda(e^{-t_0})\Big) - \Big(-ln\Big(\lambda(e^{-t_1})\Big)\Big)\Big|.$$

We can avoid the absolute value, since $\lambda : [0,1] \to [0,1]$ is non-decreasing: $t_1 > t_0 \Rightarrow e^{-t_1} < e^{-t_0} \Rightarrow \lambda(e^{-t_1}) \leq \lambda(e^{-t_0}) \Rightarrow ln(\lambda(e^{-t_1})) \leq ln(\lambda(e^{-t_0}))$ Hence,

$$d(\hat{f}(x), \hat{f}(x')) = ln(\lambda(e^{-t_1})) - ln(\lambda(e^{-t_0})). \tag{2}$$

The convexity of λ will allow us to relate this distance with $t_1 - t_0$. The idea is that if we have two points on the line $y = Kx$ $(y_1 = Kx_1,\ y_2 = Kx_2)$, the difference between the logarithms only depends on the proportion between x_1 and x_2 since $ln(Kx_1) - ln(Kx_2) = ln(\frac{Kx_1}{Kx_2}) = ln(\frac{x_1}{x_2})$ and in our case, this proportion between two points in the image of λ may be bounded using the line which joins the $(0,0)$ with the first point since λ is convex.

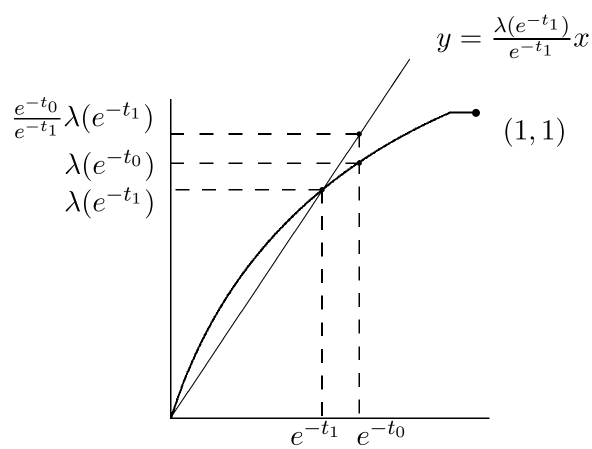

Figure 4: The function λ is convex.

Since $\lambda : [0,1] \to [0,1]$ is convex and $e^{-t_1} < e^{-t_0}$, we have that $\lambda(e^{-t_0}) \leq \frac{e^{-t_0}}{e^{-t_1}}\lambda(e^{-t_1}) \Rightarrow$ since the natural logarithm is an increasing function, substituting in (2),

$$d(\hat{f}(x), \hat{f}(x')) \leq ln(\frac{e^{-t_0}}{e^{-t_1}}\lambda(e^{-t_1})) - ln(\lambda(e^{-t_1})) = ln(e^{t_1 - t_0}) = t_1 - t_0 = d(x, x').$$

Case II. Suppose that $x,\ x'$ are not in the same branch. Then there exist $F,\ G \in end(T,v)$ and $t_0, t_1 \in R$ such that $x = F(t_0),\ x' = G(t_1)$ and let $t_2 = sup\{s |\ F(s) = G(s)\}$. Then $t_2 \leq t_0, t_1$ (if it was not then x and x' would be in the same branch) and $d(x,x') = t_0 - t_2 + t_1 - t_2 = d(x,y) + d(y,x')$ with $y = F(t_2) = G(t_2)$.

Nevertheless, $\hat{f}(F(t_2)) = \hat{f}(y) = \hat{f}(G(t_2))$ and by case I, we can see that $d(\hat{f}(x), \hat{f}(x')) \leq d(\hat{f}(x), \hat{f}(y)) + d(\hat{f}(y), \hat{f}(x')) \leq d(x,y) + d(y,x') = d(x,x')$. □

Remark 7.1. *Being Lipschitz, the induced map $\hat{f}$ is uniformly continuous.*

Metrically proper between trees

Proposition 7.1.12. *If f is a uniformly continuous map between the end spaces then $\hat{f}$ is metrically proper between trees.*

Proof. We have already proved the continuity.

Rooted. We assumed $\lambda(1) = 1$ and the image of the root will be the image of $F(0)$ for any $F \in end(T,v)$, thus

$$\hat{f}(v) = \hat{f}(F(0)) = f(F)\Big(- ln(\lambda(e^0))\Big) = f(F)(0) = w.$$

Metrically proper. We need to show that $\forall M > 0 \quad \exists N > 0$ such that $\hat{f}^{-1}(B(w,M)) \subset B(v,N)$.

(This is equivalent to say that the inverse image of bounded sets is bounded.).

$\hat{f}^{-1}(B(w,M)) = \{x \in T |\ - ln(\lambda(e^{-\|x\|})) < M\}$. By remark 7.1.9 $-ln(\lambda(e^{-t}))$ is non-decreasing and by remark 7.1.10 it is clear that $\exists N > 0$ such that $\forall t \geq N \quad - ln(\lambda(e^{-t})) > M$, and hence, $\hat{f}^{-1}(B(w,M)) \subset B(v,N)$. □

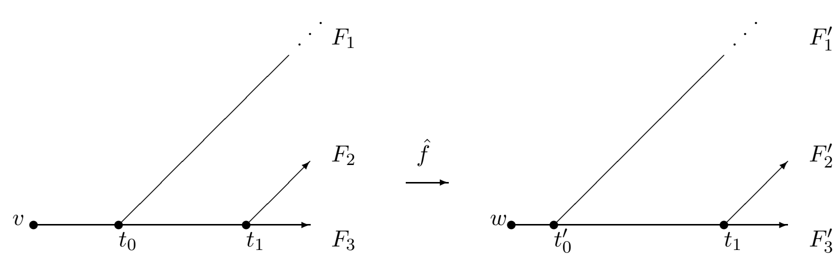

Figure 5: $\hat{f} : (T,v) \to (T',w)$ induced by a map f between the ends.

Example 7.1.13.

Let $f : end(T, v) \to end(T', w)$ such that $f(F_i) = F'_i$ for $i = 1, 2$ or 3. A modulus of continuity can be defined as in lemma 7.1.2

$$\varrho(\delta) := \begin{cases} 0 & \text{if } \delta < e^{-t_1}, \\ e^{-t_1} & \text{if } e^{-t_1} \leq \delta < e^{-t_0}, \\ e^{-t'_0} & \text{if } e^{-t_0} \leq \delta < 1, \\ 1 & \text{if } 1 \leq \delta. \end{cases}$$

Now, if we construct ω as in 7.1.5, we have

$$\omega(\delta) := \begin{cases} \frac{e^{-t'_0}}{e^{-t_0}} \cdot \delta & \text{if } \delta < e^{-t_0}, \\ \varrho_{e^{-t_0},1}(\delta) & \text{if } e^{-t_0} \leq \delta < 1, \\ 1 & \text{if } 1 \leq \delta. \end{cases}$$

Making $\lambda := \omega|_{[0,1]}$

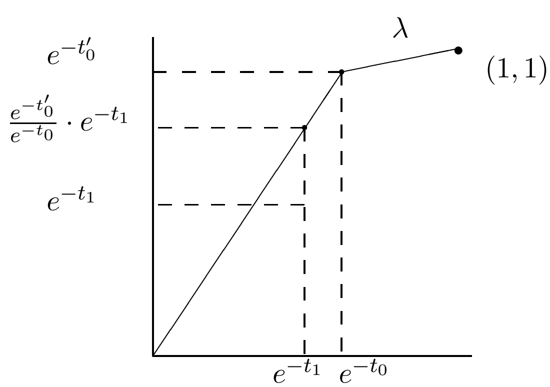

Figure 6: λ can be defined as follows.

We can check that $\hat{f}$ is Lipschitz of constant ≤ 1 from $F_i[0, t_0]$ to $F'_i[0, t'_0]$ and an isometry between $F_i[t_0, \infty)$ and $F_i[t'_0, \infty)$ for $i = 1, 2$ or 3 with $\hat{f}(F_i(t_0)) = F'_i(t'_0)$ and $\hat{f}(F_j(t_1)) = F'_j(t_1 - t_0 + t'_0) \in F'_j(t'_0, t_1)$ for $j = 2, 3$. Thus, f is a non-expansive map.

Some remarks

Definition 7.1.14. *Let X_1, X_2 two metric spaces, a map $f : X_1 \to X_2$ is* bornologous *if for every $R > 0$ there is $S > 0$ such that for any two points $x, x' \in X_1$ with $d(x, x') < R$, $d(f(x), f(x')) < S$.*

Definition 7.1.15. *The map is* coarse *if it is metrically proper and bornologous.*

Remark 7.1.16. *The induced map between the trees $\hat{f}$ from the uniformly continuous map between the end spaces is coarse.*

Proof. We have already seen that it is metrically proper. Since it is lipschitz of constant 1, then $\forall S > 0 \quad \exists R > 0$ such that $\quad d(x, x') < S \Rightarrow d(\hat{f}(x), \hat{f}(x')) < R$, with only making $R = S$. $\square$

Definition 7.1.17. *A map is* proper *if the inverse image of any compact set is compact.*

We studied if $\hat{f}$ is also proper but it isn't.

Counterexample. Let U a ultrametric space consisting of a countable family, non finite, of points $\{x_n\}_{n\in\mathbb{N}}$ with $d(x_i, x_j) = d_1 \ \forall i \neq j$ and another point, $\{y\}$ with $d(y, x_i) = d_0 \ \forall i$, suppose $d_0 > d_1$, and let U' the same family of points $\{x_n\}_{n\in\mathbb{N}}$ with distance d_1 among them and another point, $\{y'\}$ with $d(y', x_i) = d'_0$ and $d'_0 > d_0$. Both spaces are uniformly discrete and the map f which sends y to y', and x_i to x'_i is obviously uniformly continuous. Now we can find a compact set K in $T_{U'}$ such that its inverse image under $\hat{f} : T_U \to T_{U'}$ is not compact.

Consider $t_0 = -ln(d_0)$ $t'_0 = -ln(d'_0)$ and $t_1 = -ln(d_1)$. The induced trees are,

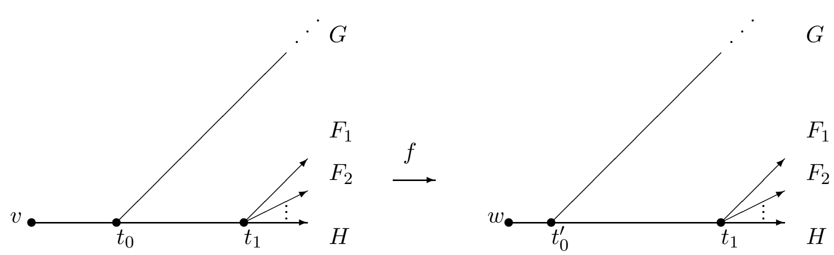

Figure 7: A metrically proper map between the trees which is not proper.

Let $K = \bar{B}(w, t_1)$ which is obviously compact, we can see that $\hat{f}^{-1}(K)$ is not compact.

The image by $\hat{f}$ of the arc $[v, x_i(t_0)] \approx [0, t_0]$ will be $[w, x'_i(t'_0)] \approx [0, t'_0]$ (with $t'_0 < t_0$). By convexity of λ, $\forall t > t_0 \quad e^{-t} < e^{-t_0} \Rightarrow \lambda(e^{-t}) \geq \frac{e^{-t}}{e^{-t_0}}\lambda(e^{-t_0}) \Rightarrow -ln(\lambda(e^{-t})) \leq -ln(e^{t_0-t} \cdot \lambda(e^{-t_0})) = t - t_0 + t'_0$. Let $\epsilon = t_0 - t'_0 > 0$ then $\hat{f}(B(v,t)) \subset B(w, t-\epsilon) \Rightarrow$ in particular $B(v, t_1+\epsilon) \subset \hat{f}^{-1}(B(w, t_1))$, and so the inverse image by $\hat{f}$ of K is a closed ball of radius greater than t_1, and since T_U is not locally compact at t_1, this set is not compact.

7.2 Uniformly continuous map between end spaces induced by a metrically proper map between trees

Proposition 7.2.1. *Given (T,v) and (T',w) two geodesically complete rooted $\mathbb{R}$-trees, and f a metrically proper map between trees then $\forall F \in end(T,v)$ $\exists!\ G \in end(T',w)$ such that $G[0,\infty) \subset im\Big(\hat{f}(F[0,\infty))\Big)$. Thus, f induces a map between the end spaces of the trees.*

Proof. <u>Existence</u>. Let $F \in end(T,v)$. $\forall n \in \mathbb{N}$, $\exists t_n > 0$ such that $\hat{f}^{-1}(B(w,n)) \subset B(v,t_n)$. By proposition 3.5 $\exists!\ c'_n \in \partial B(w,n)$ such that $f\Big(T_{F(t_n)}\Big) \subset T'_{c'_n}$.

Define $G : [0,\infty) \to T$ such that $G|_{[0,n]} \equiv [w, c'_n]$ $\forall n \in \mathbb{N}$. It is clear that this G is well defined, $G \in end(T,v)$ and $G[0,\infty) \subset im\Big(\hat{f}(F[0,\infty))\Big)$ q.e.d.

<u>Uniqueness</u>: Let $H \in end(T',w)$ $H \neq G$ with $d(H,G) = d_0 > 0$ we are going to show that $H[0,\infty)$ can't be contained in the image of $F[0,\infty)$ by f.

Figure 8: Uniqueness.

Let $M > -ln(d_0)$. As we know, $\exists N > 0$ such that $\hat{f}^{-1}(B(w,M)) \subset B(v,N)$. By proposition 3.5 $\exists!\ c'_M \in \partial B(w,M)$ such that $f\Big(T_{F(N)}\Big) \subset T'_{c'_M}$ and it is clear that $c'_M = G(M)$ but since $M > -ln(d_0) = sup\{s/G(s) = H(s)\} \Rightarrow H(M) \neq c'_M \Rightarrow \hat{f}(F[N,\infty)) \cap H[0,\infty) = \emptyset$.

Moreover (T,v), (T',w) are metric spaces and $\hat{f}$ is continuous, then $\hat{f}(F[0,N])$ is the continuous image of a compact set and so it is compact in a metric space and hence, it is bounded $\Longrightarrow H[0,\infty) \not\subset \hat{f}(F[0,N])$.

Hence $H[0,\infty) \not\subset \hat{f}(F[0,\infty))$ and G is unique. $\square$

Definition 7.2.2. *Let $f : (T,v) \to (T',w)$ a metrically proper map between trees, define $\tilde{f} : end(T,v) \to end(T',w)$ with $\tilde{f}(F) = G \in end(T',w)$ such that $G[0,\infty) \subset f(F[0,\infty))$.*

Proposition 7.2.3. *$\tilde{f}$ is uniformly continuous.*

Proof. Let $\epsilon' > 0$. Consider $\epsilon < \epsilon'$. Then there exists $\delta > 0$ such that $\hat{f}^{-1}(B(w,-ln\epsilon)) \subset B(v,-ln\delta) \Rightarrow \hat{f}(T \backslash B(v,-ln\delta)) \subset T' \backslash B(w,-ln\epsilon)$. Once again, the idea of 3.5.

Consider two branches F and G on (T,v) (two elements in the end space of the tree) with $d(F,G) \leq \delta$, this means that $F(t) = G(t)$ on $[0, -ln\delta]$. Let $c = F(-ln\delta) = G(-ln\delta)$, we have seen that $f(c) \in T' \backslash B(w, -ln\epsilon)$, then $\tilde{f}(F) = \tilde{f}(G)$ at least on $[0, -ln\epsilon]$, thus $d(\hat{f}(F), \hat{f}/G)) \leq \epsilon < \epsilon'$ and hence $\tilde{f}$ is uniformly continuous. $\square$

Proposition 7.2.4. *Let $f, f' : (T,v) \to (T', w')$ two metrically proper maps between trees, $f \sim f' \Leftrightarrow \tilde{f} = \tilde{f}'$ (this is, if they induce the same map between the end spaces).*

Proof. Suppose $f \sim f'$ and they don't induce the same map. $\exists F \in end(T,v)$ such that $\tilde{f}(F) = G \neq H = \tilde{f}'(F)$. Let $M > -ln(d(G,H)) > 0$, $N_0 > 0$ such that $f^{-1}(B(w,M)) \subset B(v, N_0)$ and $f'^{-1}(B(w,M)) \subset B(v,N_0)$ then, $\forall N > N_0$, let $c = F(N) \in \partial B(v,N)$ and by 7.2.1 $f_{\mathcal{T}_N}(T_c) = T'_{G(M)} \neq T'_{H(M)} = f'_{\mathcal{T}_N}(T_c)$ which are different because $M > -ln(d(G,H))$ which is a contradiction with $(f \sim f')$.

Conversely, suppose that f and f' induce the same map between the end spaces. Since they are metrically proper $\forall M > 0$ $\exists N_1 > 0$ such that $f(T \backslash B(v, N_1)) \subset T' \backslash B(w,M)$ and $\exists N_2 > 0$ such that $f(T \backslash B(v,N_2)) \subset T' \backslash B(w,M)$. Let $N_0 = \max\{(N_1, N_2)\} \Rightarrow f(T\backslash B(v,N_0)) \subset T' \backslash B(w,M)$ and $\forall N > N_0$, we have two maps as we saw in 3.6.

$$f_{\mathcal{T}_N}, f'_{\mathcal{T}_N} : \mathcal{T}_N \longrightarrow \mathcal{T}'_M.$$

The induced map between the end spaces is the same, hence $\forall F \in end(T,v)$ $\exists!\, G \in end(T',w)$ such that $\tilde{f}(F) = G = \tilde{f}'(F)$. Consider $T_{F(N)}$ any subtree of $T \backslash B(v,N)$, and it is clear that the image of $F[N,\infty)$ whether by f or f' must be contained in $T'_{G(M)}$ since $G[0,\infty)$ is contained in the image of $F[0,\infty)$. Thus $f_{\mathcal{T}_N}(T_{F(N)}) = T'_{G(M)} = f'_{\mathcal{T}_N}(T_{F(N)}) \Longrightarrow f \sim f'$. $\square$

Corollary 7.2.5. *Let $f, f' : (T,v) \to (T', w')$ two metrically proper maps between trees, $f \simeq_{Mp} f' \Leftrightarrow \tilde{f} = \tilde{f}'$*

Corollary 7.2.6. *In any equivalence class of metrically proper maps between the trees there is a representative which is Lipschitz of constant 1 and that restricted to the complement of some open ball centered on the root, the restriction to the branches is injective.*

Remark 7.2.7. *Given $f : (T,v) \to (T',w')$ a surjective metrically proper map between trees, arise the question if the induced map between the end spaces would also be surjective. It is not.*

Counterexample.

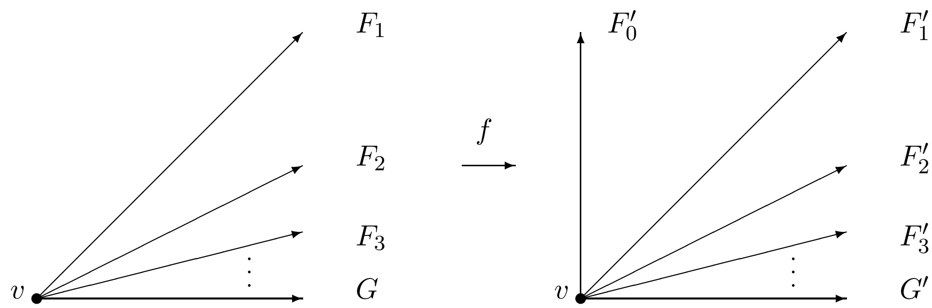

Figure 9: A surjective metrically proper map between the trees which induces a non surjective map between the ends.

Let

$$f(F_n(t)) = \begin{cases} F'_0(t) & \text{if } \; t \in [0, \frac{1}{4}], \\ F'_0(\frac{1}{4} + 4n(t - \frac{1}{4})) & \text{if } \; t \in [\frac{1}{4}, \frac{1}{2}], \\ F'_0(2n(1-t)) & \text{if } t \in (\frac{1}{2}, 1], \\ F'_n(t-1) & \text{if } t \in (1, \infty). \end{cases}$$

and

$$f(G(t)) = G'(t).$$

f is clearly rooted, continuous, surjective and metrically proper but if we consider the induced map between the end spaces we find that F'_0 is not contained in the image of any branch of T.

8 Equivalence of categories

Consider the categories,

$\mathcal{T}$: Geodesically complete rooted $\mathbb{R}$-trees and metrically proper homotopy classes of metrically proper maps between trees.

$\mathcal{U}$: Complete ultrametric spaces of diameter ≤ 1 and uniformly continuous maps. Define the functors,

$\xi : \mathcal{T} \longrightarrow \mathcal{U}$ such that $\xi(T, v) = end(T, v)$ for any geodesically complete rooted $\mathbb{R}$-tree and $\xi([f]_{Hp}) = \tilde{f}$ for any metrically proper homotopy class of a metrically proper map between trees.

$\eta : \mathcal{U} \longrightarrow \mathcal{T}$ such that $\eta(U) = T_U$ for any complete ultrametric space of diameter ≤ 1 and $\eta(f) = [\hat{f}]$ for any uniformly continuous map.

Proposition 8.1. $\xi : \mathcal{T} \longrightarrow \mathcal{U}$ *is a functor.*

Proof. $\xi(id_{(T,v)}) = id_{end(T,v)}$ is obvious.

Let $[f] : (T,v) \to (S,w)$, $\quad [g] : (S,w) \to (R,z)$ two equivalence classes of metrically proper maps between trees then

$$\xi([g] \circ [f]) = \xi([g]) \circ \xi([f]).$$

By 7.2.1, the induced maps between the end spaces are clearly the same. $\square$

Proposition 8.2. *$\eta : \mathcal{U} \longrightarrow \mathcal{T}$ is a functor.*

Proof. $\eta(id_U) = \eta(id_{end(T_U)}) = id_{T_U}$ is obvious.

Let $f : U_1 \to U_2$, and $g : U_2 \to U_3$ two uniformly continuous maps then

$$\eta(g \circ f) = \eta(g) \circ \eta(f).$$

This follows immediately from 7.2.4 since the maps between the end spaces are the same. $\square$

Lemma 8.3. *Let $S : A \to C$ be a functor between two categories. S is an equivalence of categories if and only if is $full$, $faithful$ and each object $c \in C$ is isomorphic to $S(a)$ for some object $a \in A$.*

This lemma is in [6]

Theorem 8.4. ***(Main theorem)*** *$\xi : \mathcal{T} \longrightarrow \mathcal{U}$ is an equivalence of categories.*

Proof. $\underline{\xi \text{ is full}}$ (immediate $f = [\tilde{\hat{f}}] = \xi(\hat{f})$).

$\underline{\xi \text{ es faithful}}$ (this follows immediately from proposition 7.2.4).

$\forall U \in \mathcal{U} \quad \exists\, T \in \mathcal{T}$ such that $\quad \xi(T) \approx U$. (By 6.3 $\xi(T_U) \approx U$, with $\approx$ isometry). $\square$

Example 8.5. *Consider $f : (T,v) \to (T',w)$ a map between the geodesically complete rooted $\mathbb{R}$-trees*

We can easily define an homeomorphism between these trees. Let f be such that $f[n-1,n] = [1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}]$ $\forall n \in \mathbb{N}$ with $f|_{[n-1,n]}$ a similarity with constant $\frac{1}{2^n}$ on this arc, and an isometry on the rest (the vertical lines) with $f(F_n) = F'_n$ $\forall n \in \mathbb{N}$. Then it is obviously an homeomorphism but clearly not uniform since f^{-1} is not uniformly continuous.

Since f is a non-expansive map, f^{-1} is metrically proper and hence, it induces a map $\widetilde{f^{-1}}$ from $end(T',w)$ to $end(T,v)$ which is uniformly continuous but f is not metrically proper (for example $f^{-1}(B(w,1))$ is not bounded)

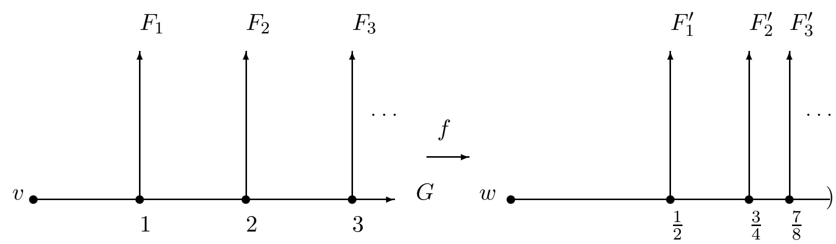

Figure 10: A homeomorphism between the trees which does not induce a map between the ends.

and it doesn't induce any map from $end(T,v)$ to $end(T',w)$ since $f(G)$ is not geodesically complete.

f is bornologous but it is not coarse (fails to be metrically proper) and f^{-1} is not bornologous.

Example 8.6. *We can define also an homeomorphism f between two rooted geodesically complete $\mathbb{R}$-trees such that $\tilde{f}$ is a non-uniform homeomorphism between the end spaces.*

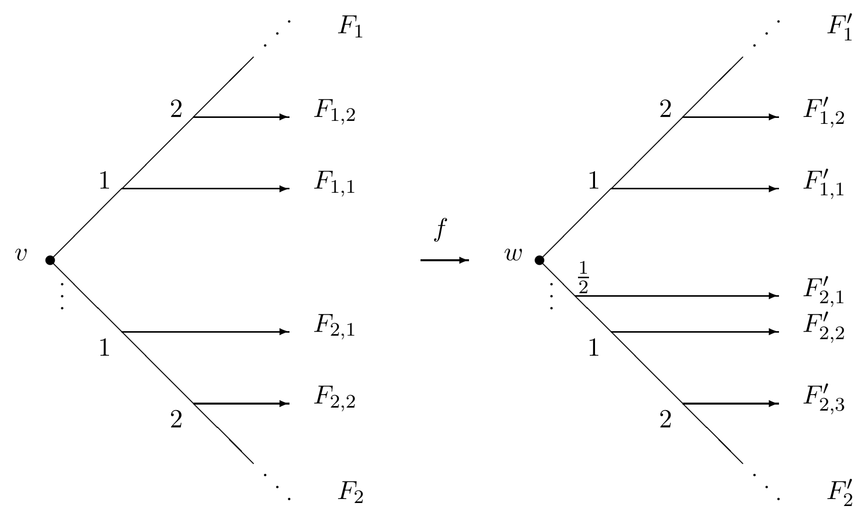

Figure 11: A homeomorphism between the trees which induces a non–uniform homeomorphism between the ends.

Consider these trees (T,v) and (T',w). (T,v) has $\{F_i\}_{i=1}^{\infty}$ branches such that $F_i \cap F_j = \{v\}$, and $\forall i$ there are branches $\{F_{i,k}\}_{k=1}^{\infty}$ such that $F_{i,k} = F_i$ on $[0,k]$. (T',w) is quite similar but $\forall i$ the branches $\{F'_{i,k}\}_{k=1}^{\infty}$ are such that $F'_{i,k} = F'_i$ on $[0,\frac{k}{i}]$ $\quad \forall k \le i$ and $F'_{i,k} = F'_i$ on $[0,k-i]$ $\quad \forall k > i$.

Define $f : (T,v) \to (T',w)$ such that $f(F_i(t)) = F'_i(\frac{t}{i})$ $\forall t \in [0,i]$ and $f(F_i(t)) = F'_i(t-i+1)$ $\forall t \in [i,\infty)$ $\forall i \in \mathbb{N}$, and also $f(F_{i,k}(t) = F'_{i,k}(t - i + \frac{k}{i})$ $\forall t \in [i,\infty)$, $\forall k \leq i$ and $f(F_{i,k}(t) = F'_{i,k}(t-i)$ $\forall t \in [i,\infty)$, $\forall k > i$. Hence, the induced map between the end spaces $\tilde{f} : end(T,v) \to end(T',w)$ is $\tilde{f}(F_i) = F'_i$ and $\tilde{f}(F_{i,k}) = F'_{i,k}$ $\forall i,k \in \mathbb{N}$. It is easy to verify that $\tilde{f}$ is an homeomorphism but this homeomorphism is not uniform. Let $\epsilon < e^{-1}$, $\forall \delta > 0$ there exists $N > 0$ such that $e^{-i} < \delta$ $\forall i \geq N$. Then, $\forall i > N$ $d(F_i, F_{i,i}) = e^{-i} < \delta$ and $d(\tilde{f}(F_i), \tilde{f}(F_{i,i})) = d(F'_i, F'_{i,i}) = e^{-1} > \epsilon$.

Define $g := \widetilde{f^{-1}}$. Then it easy to check that g is uniformly continuous and the induced map $\hat{g}$ is such that $\hat{g}|_{F'[0,\infty)} \to F[0,\infty)$ is an isometric embedding $\forall F' \in end(T',w)$.

Nevertheless, the end spaces of these trees are in fact uniformly homeomorphic, and hence, as it has been proved, there are $f : (T,v) \to (T',w)$, and $f' : (T',w) \to (T,v)$ metrically proper maps between trees such that $f \circ f' \simeq_P id_{T'}$ and $f' \circ f \simeq_P id_T$. These can be $f' = \hat{g}$ and $f : (T,v) \to (T',w)$ such that $f(F[0,1]) = w$ $\forall F \in end(T,v)$, $f(F_i(t)) = F'_i(t-1)$ $\forall t \in [1,\infty)$, $\forall i \in \mathbb{N}$, $f(F_{i,k}(t)) = F_{i,k+i-1}(t-1)$ $\forall t \in [1,\infty)$, $\forall k \geq 2$ and finally $f(F_{\frac{i\cdot(i-1)}{2}+k,1}) = F'_{i,k}$ $\forall k \leq i$, $\forall i \in \mathbb{N}$. The uniform homeomorphism is the naturally induced by these maps.

9 Lipschitz maps and coarse maps between trees

In this section lipschitz may be understood directly as non-expansive, or lipschitz of constant 1. Nevertheless what is written is true in general for the usual definition of lipschitz.

Lemma 9.1. *Let x_1, x_2, y_1, y_2 points in $\mathbb{R}$ then for any $t \in [0,1]$,*

$$d(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \leq max\{d(x_1,y_1), d(x_2,y_2)\}$$

Proof. $d(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) = |tx_1 + (1-t)x_2 - [ty_1 + (1-t)y_2]| = |t(x_1 - y_1) + (1-t)(x_2 - y_2)| \leq t \cdot |x_1 - y_1| + (1-t) \cdot |x_2 - y_2| \leq max\{d(x_1,y_1), d(x_2,y_2)\}$. □

Lemma 9.2. *Let f,$g : T \to T'$ two metrically proper maps between trees. Consider $H : T \times I \to T'$ the shortest path homotopy defined in lemma 3.10, then for any two points $x, y \in T$,*

$$d(H_t(x), H_t(y)) \leq max\{d(f(x), f(y)), d(g(x), g(y))\}.$$

Proof. Suppose $d(f(x), f(y)) < d(g(x), g(y))$. If for some $t \in I$ $d(H_t(x), H_t(y)) > d(g(x), g(y))$ then there must be some $t_0 > t \in I$ such that $d(H_{t_0}(x), H_{t_0}(y)) = d(g(x), g(y))$. So let us assume $d(f(x), f(y)) = d(g(x), g(y)) = d_0$, and it suffices to show that in this case the condition is satisfied.

Now if we show that in this conditions there is always some $\epsilon > 0$ such that for any $0 < t < \epsilon \quad d(H_t(x), H_t(y)) \leq d_0$ then we have that this happens for any t in an open set of I and by continuity of the metric, this will be also a closed set of I and hence, $d(H_t(x), H_t(y)) \leq d_0 \ \forall t \in I$.

Now to prove the lemma it suffices to distinguish the following cases.

Case 1. If $f(x) = g(x)$ (or $f(y) = g(y)$). Then there is a unique arc, isometric to certain interval in $\mathbb{R}$ that contains the points and this is the case of lemma 9.1.

Now we can assume $f(x) \neq g(x)$ and $f(y) \neq g(y)$.

Case 2. If $f(x) \notin [w, g(x)]$ and $f(y) \notin [w, g(y)]$. Then there exists $\delta > 0$ such that $\delta < d(f(x), [w, g(x)])$ and $\delta < d(f(y), [w, g(y)])$. Let ϵ such that $\epsilon < \frac{\delta}{d(f(x),g(x))}$ and $\epsilon < \frac{\delta}{d(f(y),g(y))}$ then for every $0 < t < \epsilon$ $H_t(x) \notin [w, g(x)] \Rightarrow H_t(x) \in [w, f(x)]$ and $H_t(y) \notin [w, g(y)] \Rightarrow H_t(y) \in [w, (f(y)]$ and it is easy to check that for every $0 < t < \epsilon$ $d(H_t(x), H_t(y)) < d(f(x), f(y)) = d_0$.

Case 3. If $f(x) \in [w, g(x)]$ and $g(x) \in [w, f(y)]$. Let $z \in T$ such that $[w, g(x)] \cap [w, f(y)] = [w, z]$. If $z = g(x)$ or $z = f(y)$ then there is an arc, isometric to an interval in $\mathbb{R}$ that contains the points and this is again the case of lemma 9.1. Now we find two different situations.

a) $z \in [w, f(x)]$ and $z \in [w, g(y)]$ then this is again an arc isometric to an interval.

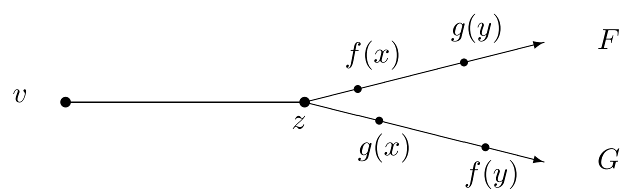

Figure 12: This is again the case of the previous lemma.

b) $z \notin [w, f(x)]$ (if $z \notin [w, g(y)]$ it is analogous). Note that both $z, f(x) \in [w, g(y)]$ and so in this case $f(x) \in [w, z]$. Let $\delta > 0$ such that $\delta < d(f(x), z)$ and $\delta < d(z, f(y))$. Let $\epsilon > 0$ such that $\epsilon < \frac{\delta}{d(f(x),g(x))}$ and $\epsilon < \frac{\delta}{d(f(y),g(y))}$ then for every $0 < t < \epsilon$ $H_t(x) \in [f(x), z]$ and $H_t(y) \in [z, f(y)]$ and hence, $d(H_t(x), H_t(y)) < d(f(x), f(y)) = d_0$.

$\square$

Define $f \simeq_L f'$ if there exists $H : T \times I \to T'$ a rooted metrically proper homotopy of f to f' such that H_t is Lipschitz for every $t \in I$.

Also, $f \simeq_C f'$ if there exists $H : T \times I \to T'$ a rooted (metrically proper) homotopy of f to f' such that H_t is coarse for every $t \in I$. Being metrically

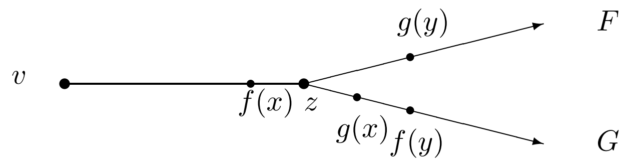

Figure 13: Case 3 b)

proper is already supposed by definition of coarse.

The next propositions follow immediately from the lemma and proposition 7.2.4.

Proposition 9.3. *Given $f, f' : T \to T'$ two lipschitz, metrically proper maps between trees, then $\tilde{f} = \tilde{f}' \Leftrightarrow f \simeq_L f'$.*

Corollary 9.4. *There is an equivalence of categories between $\mathcal{U}$ and the category of geodesically complete rooted $\mathbb{R}$-trees with lipschitz, metrically proper homotopy classes of lipschitz, metrically proper maps between trees.*

Proposition 9.5. *Given $f, f' : T \to T'$ two coarse, metrically proper maps between trees, then $\tilde{f} = \tilde{f}' \Leftrightarrow f \simeq_C f'$.*

Corollary 9.6. *There is an equivalence of categories between $\mathcal{U}$ and the category of geodesically complete rooted $\mathbb{R}$-trees with coarse, (metrically proper) homotopy classes of coarse, (metrically proper) maps between trees.*

Corollary 9.7. *Given $f : T \to T'$ a metrically proper map between trees then there exists a rooted continuous metrically proper non-expansive map $f' : T \to T'$ such that $f \simeq_{Mp} f'$.*

Corollary 9.8. *Given $f : T \to T'$ a rooted continuous coarse map between trees then there exists a rooted continuous metrically proper non-expansive map $f' : T \to T'$ such that $f \simeq_C f'$.*

10 Freudenthal ends and classical results

This work allows us to give some new proofs of already known results and to look at them from a new perspective. We also extend in this section the field of our study to include some considerations about non-rooted and non-geodesically complete trees and how can we use or adapt our tools with them.

Pruning the tree When we have a non-geodesically complete rooted $\mathbb{R}$-tree and we are only interested in the geodesically complete branches we can prune the rest as follows.

Theorem 10.1. *If (T, v) is a rooted $\mathbb{R}$-tree then, there exists $(T_\infty, v) \subset (T, v)$ a unique geodesically complete subtree that is maximal.*

Proof. Using Zorn's lemma. Consider $(\mathcal{T}_{gc}, \leq)$ with $\mathcal{T}_{gc}$ geodesically complete subtrees of (T, v) and $T_1 \leq T_2 \Leftrightarrow T_1 \subset T_2$. This is an ordered structure.

It is not empty since the root is a trivial geodesically complete subtree.

To prove that every chain of $(\mathcal{T}_{gc}, \leq)$ admits an upper bound T_M it suffices to show that the union of elements of the chain is also a geodesically complete subtree of (T, v). It is a subset of the tree where every point is arcwise connected to the root and hence it is obviously a subtree. Let $f : [0, t] \to T_M, t > 0$ any isometric embedding such that $f(0) = v$, then there exists an element T_0 in the chain such than $f(t) \in T_0 \Rightarrow f[0, t] \in T_0$ and f extends to an isometric embedding $\tilde{f} : [0, \infty) \to T_0 \subset T_M$, and hence, T_M is geodesically complete.

Then, by Zorn's lemma $(\mathcal{T}_{gc}, \leq)$ possesses a maximal element.

The union of two elements of $(\mathcal{T}_{gc}, \leq)$ is also a geodesically complete subtree and hence, the maximal element (T_∞, v) is unique. □

Lemma 10.2. *If the metric of (T_∞, v) is proper then it is a deformation retract of (T, v).*

Proof. Since the metric is proper, for any $x \in T \backslash T_\infty$ there is a point $y \in T_\infty$ such that $d(x, T_\infty) = d(x, y)$ and it is unique since the tree is uniquely arcwise connected. Let $r : T \to T_\infty$ such that $r(x) = y \ \forall x \in T \backslash T_\infty$ and the identity on T_∞. Then r is a retraction and the shortest path homotopy makes the deformation retract. □

Proper homotopies and Freudenthal ends

Definition 10.3. *Two proper maps $f, g : X \to Y$ are properly homotopic $f \simeq_p g$ in the usual sense if there exists an homotopy $H : X \times I \to Y$ of f to g such that H is proper.*

Definition 10.4. *X, Y are of the same proper homotopy type or properly homotopic in the usual sense if there exist two proper maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq_p Id_X$ and $f \circ g \simeq_p Id_Y$.*

Notation: $\simeq_P$ means properly homotopic such that the proper maps and the homotopy are rooted, and $\simeq_p$ is the usual sense of proper homotopy equivalence.

Lemma 10.5. *Let S_1, S_2 two locally finite simplicial trees and consider any two points $x_1 \in S_1, x_2 \in S_2$. Then $(S_1, x_1) \simeq_P (S_2, x_2)$ if and only if $S_1 \simeq_p S_2$.*

Proof. The only if part is clear since it is a particular case.

The other part is rather technical. Consider $f : S_1 \to S_2$ and $g : S_2 \to S_1$ proper maps and the proper homotopies H^1 of $g \circ f$ to Id_{S_1} and H_2 of $f \circ g$ to Id_{S_2}. First we construct two rooted proper maps redefining f and g. Consider the unique arc in S_2 $[x_2, f(x_1)]$. In order to define the rooted proper map from S_1 to S_2 we are going to send this arc with a proper homotopy to the root x_2 and to pull somehow the rest of the tree after it.

Since $[x_2, f(x_1)]$ is compact and the tree is locally finite, there are finitely many vertices $v_1, \ldots, v_n$ in this arc. The tree is locally compact, hence consider $\overline{B}(v_i, \epsilon_i)$ compact neighborhoods of v_i with $i = 1, \ldots, n$ (we may assume that they are disjoint). We define the homotopy such that sends $[x_2, f(x_1)]$ to x_2, that for each point $y \in T_{v_i} \cap \partial\overline{B}(v_i, \epsilon_i)$ goes linearly from $[v_i, y]$ to $[x_2, y]$ and is the identity on the rest as follows. If $x \in [x_2, f(x_1)]$ let $j_x : [0, d(x_2, x)] \to [x_2, x]$ an isometry with $j_x(0) = x$ then $H(x,t) = j_x(t \cdot d(x_2, x))$. If $x \in T_{v_i} \cap \overline{B}(v_i, \epsilon_i)$ then let $j_x : [0, d(x_2, x)] \to [x_2, x]$ an isometry such that $j_x(0) = x$ then $H(x,t) = j_x\big(t \cdot \big[\frac{d(v_i,x_2)+\epsilon}{\epsilon}\big(\epsilon - d(x, v_i)\big) - (\epsilon - d(x,v))\big]\big) = j_x\big(t \cdot \frac{d(v_i,x_2)}{\epsilon}(\epsilon - d(x,v))\big)$.

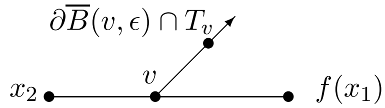

Figure 14: The homotopy sends $[x_2, f(x_1)]$ to x_2 and $[v, \partial\overline{B}(v,\epsilon) \cap T_v]$ to $[x_2, \partial\overline{B}(v,\epsilon) \cap T_v]$.

It is easy to check that $H(v_i \times I) = [x_2, v_i]$ with $H(v_i, 0) = v_i$ and $H(v_i, 1) = x_2$, and $\forall y \in \partial\overline{B}(v_i, \epsilon_i) \cap T_{v_i} \quad H(y,t) = y \quad \forall t$. $H(x,t) = x$ on the rest of the tree. This map is continuous. To see that it is proper first consider $K_0 := [x_2, f(x_1)] \cup (\bigcup_{i=1}^{n} \overline{B}(v_i, \epsilon_i))$ which is a compact subset of the tree S_2, and hence $K_0 \times I$ is a compact subset of $S_2 \times I$. For any compact set $K \in S_2$ $H^{-1}(K)$ is a closed (since H is continuous) subset of the compact set $K_0 \cup K$. Thus, H is proper.

Clearly $f(x) = H(x, 0)$ and let $\tilde{f}(x) := H(x, 1)$. $\tilde{f}$ is proper, $\tilde{f}(x_1) = x_2$ (it is rooted) and $f \simeq_P \tilde{f}$.

We do the same for $g : S_2 \to S_1$ and we get a rooted proper map $\tilde{g} : (S_2, x_2) \to (S_1, x_1)$ such that $g \simeq_P \tilde{g}$.

Hence we have a proper homotopy H^1 of $\tilde{g} \circ \tilde{f}$ to Id_{S_1} and also H^2 of $\tilde{f} \circ \tilde{g}$ to Id_{S_2}.

H^1 is such that $H^1(x_1, 0) = x_1 = H^1(x_1, 1)$. S_1 is locally compact since it is locally finite, thus, consider $\overline{B}(x_1, \epsilon)$ a compact neighborhood of the root, and so $\overline{B}(x_1, \epsilon) \times I$ is compact. Now we define the rooted homotopy which is the same at levels 0,1 ($\tilde{H}^1(x, 0) = H^1(x, 0)$ and $\tilde{H}^1(x, 1) = H^1(x, 1)$ $\forall x \in S_1$)

and in the complement of the closed (and compact) ball, (for each $t \in (0,1)$, $\tilde{H}^1(S_1 \backslash \overline{B}(x_1,\epsilon),t) = H^1(S_1 \backslash \overline{B}(x_1,\epsilon),t)$). In the closed ball we only need $\tilde{H}^1(x_1,t) = x_1 \forall t$ and $\tilde{H}^1(x,t) = H(x,t) \forall x \in \partial\overline{B}(x_1,\epsilon) \forall t$. We can define that homotopy such that $\tilde{H}^1(x,t) \subset H^1(\overline{B}(x_1,\epsilon) \times I)$ $\forall x \in \overline{B}(x_1,\epsilon)$ and since $H^1(\overline{B}(x_1,\epsilon) \times I)$ is compact, $\tilde{H}$ is also proper and rooted. We do the same with H^2 and finally we have that $(S_1,x_1) \simeq_P (S_2,x_2)$. □

We can now give another proof of the following corollary in [1].

Proposition 10.6. *Two locally finite simplicial trees are properly homotopic (in the usual sense) if and only if their Freudenthal ends are homeomorphic.*

Proof. Let S_1, S_2 two simplicial, locally finite trees. Let $v \in S_1$ and $w \in S_2$ any two points, hence (S_1,v) and (S_2,w) are two rooted trees, and by lemma 10.5 $(S_1,v) \simeq_P (S_2,w)$ if and only if $S_1 \simeq_p S_2$.

We can change the metric on the simplices and assume length 1 for each simplex. Then we have two homeomorphic copies of the simplicial rooted trees $(S'_1,v) \cong (S_1,v)$ and $(S'_2,w) \cong (S_2,w)$ (in particular $(S'_1,v) \simeq_P (S_1,v)$ and $(S'_2,w) \simeq_P (S_2,w)$), such that the non-compact branches are geodesically complete.

The metrics on (S'_1,v) and (S'_2,w) are proper. It suffices to check that any closed ball centered at the root is compact and this can be easily done by induction on the radius. Since the trees are locally finite and the distance between two vertices is at least 1, the closed ball $\overline{B}(v,1)$ (similarly $\overline{B}(w,1)$) is a finite union of compact sets (isometric to the subinterval $[0,1]$ in $\mathbb{R}$). Let $\overline{B}(v,n)$ a finite union of compact sets, $\partial\overline{B}(v,n)$ is a finite number of vertices and, since the trees are locally finite and the distance between two vertices is at least 1, $\overline{B}(v,n+1)$ is also a finite union of compact sets. Thus every closed ball centered at the root is compact.

(S'_1,v) and (S'_2,w) are proper length spaces, and by the Hopf-Rinow theorem, see [10], (S'_1,v) and (S'_2,w) are complete and locally compact.

Now consider the maximal geodesically complete subtrees (T_1,v) and (T_2,w) of (S'_1,v) and (S'_2,w) (Note that these are $\emptyset$ if and only if (S_1,v) and (S_2,w) are compact). These trees are locally finite, complete, geodesically complete and their metrics are proper. We can now find a proper homotopy equivalence between the pruned tree T_i and S'_i. The retractions $r_i : (S'_i,v) \to (T_i,v)$, $i = 1,2$, such that $r_i(x) = y$ with $d(x,T_i) = d(x,y)$ defined in lemma 10.2 are proper maps since after the change of metric the bounded branches are compact and the tree is supposed to be locally finite. Clearly this retraction and the inclusion give us a rooted proper homotopy equivalence between the trees, $(S'_1,v) \simeq_P (T_1,v)$ and $(S'_2,w) \simeq_P (T_2,w)$. Thus

$$(S_1,v) \simeq_P (T_1,v) \text{ and } S_2,w) \simeq_P (T_2,w)$$

It is well known that in this conditions $end(T_1,v) = Fr(S'_1,v) = Fr(S_1)$ and $end(T_2,w) = Fr(S'2,w) = Fr(S_2)$ and as we proved, $end(T_1,v) \cong$

$end(T_2, w) \Leftrightarrow (T_1, v) \simeq_{Mp} (T_2, w)$. If the metric is proper $(T_1, v) \simeq_{Mp} (T_2, w) \Leftrightarrow (T_1, v) \simeq_P (T_2, w)$ and hence $Fr(S_1) \cong Fr(S_2) \Leftrightarrow (T_1, v) \simeq_P (T_2, w)$.

Thus, $Fr(S_1) \cong Fr(S_2) \Leftrightarrow (S_1, v) \simeq_P (S_2, w) \Leftrightarrow S_1 \simeq_p S_2$. □

There is also an immediate proof of the following corollary in [4].

Proposition 10.7. *Two geodesically complete rooted $\mathbb{R}$-trees, (T, v) and (S, w), are rooted isometric if and only if $end(T, v)$ and $end(S, w)$ are isometric.*

Proof. If there is an isometry between the trees then the induced map between their end spaces is clearly an isometry.

Let $f : end(T, v) \to end(S, w)$ an isometry between the end spaces. Then, to induce the map between the trees we can take the identity as modulus of continuity. If $\lambda \equiv Id_{[0,1]}$ then $f(F)(-ln(\lambda(e^{-t}))) = f(F)(t) \forall F \in end(T, v) \forall t \in [0, \infty)$ and the map restricted to the branches is an isometry. For any two points in different branches $x = F(t), y = G(t')$ with $-ln(d(F, G)) < t, t'$. Since the end spaces are isometric, the distance between two branches is the same between their images and hence $d(\hat{f}(x), \hat{f}(y)) = t + t' - 2(-ln(d(f(F), f(G)))) = t + t' - 2(-ln(d(F, G))) = d(x, y)$ and $\hat{f}$ is an isometry between the trees. □

Non-rooted maps between the trees If the map is not rooted we can extend the idea of the rooted case and define how a non-rooted metrically proper map induces a map between the end spaces.

Let $f : (T, v) \to (T', w)$ any metrically proper (non-rooted) map between two geodesically complete rooted $\mathbb{R}$-trees. Then $\forall M > 0 \quad \exists N > 0$ such that $f(B(v, N)) \subset B(f(v), M)$. Let $d_0 := d(w, f(v))$, hence $f(B(v, N)) \subset B(w, M{+}d_0)$ and this is equivalent to $f^{-1}(T' \backslash B(w, M{+}d_0)) \subset T \backslash f(B(v, N))$. Now we can induce a uniformly continuous map between the end spaces almost like in 7.2.1, since for each branch $F \in (T, v)$ there is a unique branch $F' \in (T', w)$ such that $F'[d_0, \infty) \subset f(F)$ and so we define $\tilde{f} : end(T, v) \to (T', w)$ such that $\tilde{f}(F) = F'$.

The results then are not so strong, as an example of this we can give the following proposition.

Proposition 10.8. *An isometry (non rooted) $f : (T, v) \to (S, w)$ between two geodesically complete rooted $\mathbb{R}$-trees, induces a bi-lipschitz homeomorphism between $end(T, v)$ and $end(S, w)$.*

Proof. Let $f : (T, v) \to (S, w)$ a non-rooted isometry. Consider F, G any two branches in $end(T, v)$ and let $x \in T$ such that $F[0, \infty) \cap G[0, \infty) = [v, x] \approx [0, -ln(d(F, G))] \subset \mathbb{R}$. Then $f(F[0, \infty)) \cap f(G[0, \infty)) = [f(v), f(x)] \approx [0, -ln(d(F, G))] \subset \mathbb{R}$ since f is an isometry. Let $d_0 = d(w, f(v))$, hence

$\tilde{f}(F) =: F'$ and $\tilde{f}(G) =: G'$ coincide at least on $[0, -ln(d(F,G)) - d_0]$ and at most on $[0, -ln(d(F,G)) + d_0]$ and so, $e^{ln(d(F,G)) - d_0} \leq d(F',G') \leq e^{ln(d(F,G)) + d_0}$ this is $e^{-d_0} \cdot d(F,G) \leq d(F',G') \leq e^{d_0} \cdot d(F,G) \Rightarrow \tilde{f}$ is bilipschitz. $\square$

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